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COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on ''Invariance principle for extension of hydrodynamics: Nonlinear viscosity''

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Recently, Karlin, Dukek, and Nonnenmacher [Phys. Rev. E **55**, 1573 (1997)] obtained an ordinary differential equation of first order for the viscosity factor, which was not solved by them. Here we solve their equation using two methods; Adams' method and the backward differentiation formula as implemented by the Numerical Algorithm Group library. When the reduced longitudinal rate is greater than zero, the numerical solutions (for different models) are within 4% with respect to the solution for the Maxwell model. However, if the longitudinal rate is negative, our results indicate that the equation may not provide a unique solution for the viscosity factor. [S1063-651X(98)07703-4]

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In a recent work [1], Karlin, Dukek, and Nonnenmacher obtained a nonlinear first-order differential equation for the viscosity factor $(R(g))$, g being the reduced longitudinal rate, which, if solved, provides a longitudinal rate dependence viscosity that is important for situations far from equilibrium. The differential equation is

$$(1-\gamma)g^2(1-gR)\frac{dR}{dg}+\gamma g^2R^2+[\tfrac{3}{2}+g(2-\gamma)]R-2=0, \tag{1}$$

For $\gamma\neq 1$, $g\neq 0$ and $gR\neq 1$, the differential equation can be written as

$$\frac{dR}{dg}=\frac{-\gamma g^2R^2-\left[\frac{3}{2}+g(2-\gamma)\right]R+2}{(1-\gamma)g^2(1-gR)}, \tag{2}$$

where γ varies from $\gamma=\frac{1}{2}$ (hard spheres) to $\gamma=1$ (Maxwellian molecules).

Assuming R and its derivative with respect to g are continuous and finite at $g=0$, from Eq. (2) we obtain that $R(0)=\frac{4}{3}$ and $R'(0)=-\frac{16}{9}+8\gamma/9$. Furthermore, if $R(g)=\Sigma_{k=0}{}^{n}[(D^kR)(0)g^k/k!]+O(g^{n+1})$, then Eq. (2) can be used to obtain a recurrence relation for the higher order derivatives. Such a Taylor expansion corresponds to a subseries of a ''regularization'' of the higher order gradient Chapman-Enskog expansions as obtained in Ref. [1]. A computer algebra program was done to obtain these higher order derivatives, two of them are

$$D^2R(0)=\frac{64}{9}-\frac{32\gamma}{3}+\frac{64\gamma^2}{27},$$

$$D^3R(0)=-\frac{1792}{27}+\frac{4352\gamma}{27}-\frac{2624\gamma^2}{27}+\frac{128\gamma^3}{9}.$$

There is a line of critical points $[R'(g)=0]$, which we denote by $R_{\rm CL}(g,\gamma)$, which can be obtained by equating the numerator of Eq. (2) to zero; the physical relevant case corresponds to $R(g)>0$. It can be shown that $R_{\rm CL}(g,\gamma)\to\frac{4}{3}$ as $g\to 0$; mathematical consistency then requires that either $R_{\rm CL}$ is not defined or is not continuous at 0. Also, $R_{\rm CL}(g,1)=R_M(g)$, where $R_M(g)$ is the solution to Eq. (1) for $\gamma=1$, in this case R_M does not give critical points. The con-

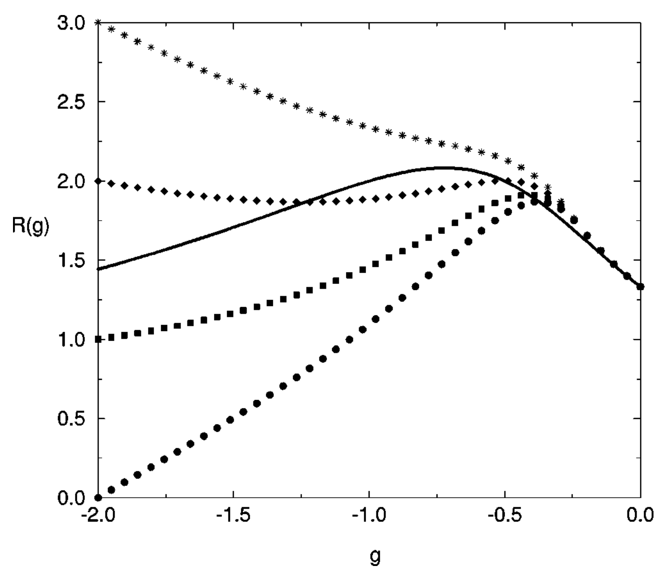

FIG. 1. Numerical solution for the viscosity factor for the rigid sphere model and different initial conditions at $g=-2.0$. The solid line corresponds to the critical line $R_{\rm CL}(g,\frac{1}{2})$.

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sideration of the critical line is useful to obtain qualitative features of the solutions. For example, in the region where R_{CL} is decreasing, if the solution is above the critical line $[R'(g)<0]$ then the function must be decreasing; furthermore, the solution cannot cross the critical line, since by continuity there would exist a region where the solution would be decreasing, but this would be a region for which $R'(g)>0$, which is a contradiction. Thus, if the solution is above the critical line then, it must be decreasing as long as R_{CL} is decreasing. It is possible to define an extension to the critical line ($R_{\mathrm{CL}}^{\mathrm{ext}}$), so that the derivatives of all orders exist and are continuous at $g=0$; furthermore, it is easy to show that the second derivative of the extension at $g=0$ is lower than $R''(0)$, implying that the solution is above the critical line for small g. The above considerations imply in particular that for $g>0$ the solution is decreasing, since the critical line is decreasing in this region. For $g<0$ the consideration of the critical again gives the possible solutions; see Fig. 1.

The differential equation (2) with initial condition $R(0)=4/3$ was solved numerically using the Adams' (AD) method and the backward differentiation formula as implemented by the NAG library. The derivative was evaluated using Eq. (2) for g outside $[-0.01,0.01]$ and using the Taylor expansion ($n=12$) otherwise. For $g\geqslant 0$ we have no problems obtaining the numerical solution for different values of γ ($\gamma\neq 1$), we found that the deviation with respect to R_M is always less than 4% for the different values of γ used (0.5, 0.6, 0.7, and 0.8). For $\gamma=0.999\,999$ the deviation, using the AD method, is less than $8\times 10^{-6}\%$. Starting with $R(0)=\frac{4}{3}$, and when integrating to negatives values of g, both methods were unable to get the solution to the requested tolerance except for small values of g. Figure 1 gives the numerical solutions for the rigid sphere model for different initial conditions at $g=-2$.

Our calculations show that for negative values of g there is no unique viscosity factor, except for small values of g where all the numerical solutions tend to an invariant set, meaning that either more information or a different differential equation are needed to obtain a unique viscosity factor for negative and non small values of g.

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[1] I. V. Karlin, G. Dukek, and T. F. Nonnenmacher, Phys. Rev. E **55**, 1573 (1997).